Polynomial-time Algorithm for Isomorphism of Graphs with Clique-width at most Three

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Abstract. The clique-width is a measure of complexity of decomposing graphs into certain tree-like structures. The class of graphs with bounded clique-width contains bounded tree-width graphs. We give a polynomial time graph isomorphism algorithm for graphs with clique-width at most three. Our work is independent of the work by Grohe et al. [16] showing that the isomorphism problem for graphs of bounded clique-width is polynomial time.

1 Introduction

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there is a bijection $f : V_1 \to V_2$ such that $\{u, v\} \in E_1$ if and only if $\{f(u), f(v)\} \in E_2$. Given a pair of graphs as input the problem of deciding if the two graphs are isomorphic is known as *graph isomorphism problem* (GI). Despite nearly five decades of research the complexity status of this problem still remains unknown. The graph isomorphism problem is not known to be in P. It is in NP but very unlikely to be NP-complete [5]. The problem is not even known to be hard for P. Recently Babai [2] designed a quasi-polynomial time algorithm to solve the GI problem improving the previously best known $2^{O(\sqrt{n \log n})}$ time algorithm [1,25]. Although the complexity of the general graph isomorphism problem remains elusive, many polynomial time algorithms are known for restricted classes of graphs e.g., bounded degree [20], bounded genus [22], bounded tree-width [3], etc.

The graph parameter *clique-width*, introduced by Courcelle et al. in [7], has been studied extensively. The class of bounded clique-width graphs is fairly large in the sense that it contains distance hereditary graphs, bounded tree-width graphs, bounded rank-width graphs [18], etc. Fellows et al. [14] shows that the computing the clique-width of a graph is NP-hard. Oum and Seymour [23] gave an elegant algorithm that computes a $(2^{3k+2} - 1)$-expression for a graph G of clique-width at most k or decides that the clique-width is more than k.

The parameters tree-width and clique-width share some similarities, for example many NP-complete problems admit polynomial time algorithms when the tree-width or the clique-width of the input graph is bounded. A polynomial time

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isomorphism algorithm for bounded tree-width graphs has been known for a long time [3]. Recently Lokhstanov et al. [19] gave an fpt algorithm for GI parameterized by tree-width. The scenario is different for bounded clique-width graphs. The complexity of GI for bounded clique-width graphs is not known. Polynomial time algorithm for GI for graphs with clique-width at most 2, which coincides with the class of co-graphs, is known probably as a folklore. The complexity of recognizing graphs with clique-width at most three was unknown until Corneil et al. [6] came up with the first polynomial time algorithm. Their algorithm (henceforth called the CHLRR algorithm) works via an extensive study of the structure of such graphs using split and modular decompositions. Apart from recognition, the CHLRR algorithm also produces a 3-expression for graphs with clique-width at most three. For fixed $k > 3$, though algorithms to recognize graphs with clique-width at most k are known [23], computing a k-expression is still open. Recently in an independent work by Grohe et al. [16] designed an isomorphism algorithm for graphs of bounded clique-width subsuming our result. Their algorithm uses group theory techniques and has worse runtime. However our algorithm has better runtime and uses different simpler intuitive techniques.

In this paper we give isomorphism algorithm for graphs with clique-width at most three with runtime $O(n^3 m)$. Our algorithm works via first defining a notion of equivalent k-expression and designing $O(n^3)$ algorithm to test if two input k-expressions are equivalent under this notion. Next we modify the CHLRR algorithm slightly to output a linear sized set $parseG$ of 4-expressions for an input graph G of clique-width at most three which runs in $O(n^3 m)$ time. Note that modified CHLRR algorithm will not output a canonical expression. However we show that for two isomorphic graphs G and H of clique-width at most three, $parseG$ contains an equivalent k-expression for each k-expression in $parseH$ and vice versa. Moreover, if G and H are not isomorphic then no pair in $parseG \times parseH$ is equivalent.

2 Preliminaries

In this paper, the graphs we consider are without multiple edges and self loops. The complement of a graph G is denoted as $\overline{G}$. The *coconnected components* of G are the connected components of $\overline{G}$. We say that a vertex v is *universal* to a vertex set X if v is adjacent to all vertices in $X \setminus \{v\}$. A *biclique* is a bipartite graph (G, X, Y), such that every vertex in X is connected to every vertex of Y. A *labeled graph* is a graph with labels assigned to vertices such that each vertex has exactly one label. In a labeled graph G, $lab(v)$ is the label of a vertex v and $lab(G)$ is the set of all labels. We say that a graph is *bilabeled* (*trilabeled*) if it is labeled using exactly two (three) labels. The set of all edges between vertices of label a and label b is denoted E_{ab}. We say E_{ab} is complete if it corresponds to a biclique.

The subgraph of G induced by $X \subseteq V(G)$ is denoted by $G[X]$, the set of vertices adjacent to v is denoted $N_G(v)$. The closed neighborhood $N_G[v]$ of v is $N_G(v) \cup \{v\}$. We write $G \cong_f H$ if f is an isomorphism between graphs

G and H. For labeled graphs G and H, we write $G \cong_f^\pi H$ if $G \cong_f H$ and $\pi : lab(G) \to lab(H)$ is a bijection such that for all $x \in V(G)$ if $lab(x) = i$ then $lab(f(x)) = \pi(i)$. The set of all isomorphisms from G to H is denoted $\mathsf{ISO}(G, H)$.

Definition 1. *The* **clique-width** *of a graph G is defined as the minimum number of labels needed to construct G using the following four operations:*

  i. $v(i)$: Creates a new vertex v with label i

  ii. $G_1 \oplus G_2 \cdots \oplus G_l$: Disjoint union of labeled graphs $G_1, G_2, \cdots, G_l$

  iii. $\eta_{i,j}$: Joins each vertex with label i to each vertex with label j $(i \neq j)$

  iv. $\rho_{i \to j}$: Renames all vertices of label i with label j

Every graph can be constructed using the above four operations, which is represented by an algebraic expression known as *k-expression*, where k is the number of labels used in expression. The *clique-width* of a graph G, denoted by $cwd(G)$, is the minimum k for which there exists a k-expression that defines the graph G. From the k-expression of a graph we can construct a tree known as *parse tree* of G. The leaves of the parse tree are vertices of G with their initial labels, and the internal nodes correspond to the operations ($\eta_{i,j}$, $\rho_{i \to j}$ and $\oplus$) used to construct G. For example, C_5 (cycle of length 5) can be constructed by

$$\eta_{1,3}((\rho_{3 \to 2}(\eta_{2,3}((\eta_{1,2}(a(1) \oplus b(2))) \oplus (\eta_{1,3}(c(3) \oplus d(1)))))) \oplus e(3)).$$

The k-expression for a graph need not be unique. The clique-width of any induced subgraph is at most the clique-width of its graph [9].

Now we describe the notions of modular and split decompositions. A set $M \subseteq V(G)$ is called a *module* of G if all vertices of M have the same set of neighbors in $V(G) \setminus M$. The *trivial modules* are $V(G)$, and $\{v\}$ for all v. In a labeled graph, a module is said to be a *l-module* if all the vertices in the module have the same label. A *prime (l-prime)* graph is a graph (labeled graph) in which all modules (*l*-modules) are trivial. The modular decomposition of a graph is one of the decomposition techniques which was introduced by Gallai [15]. The *modular decomposition* of a graph G is a rooted tree T_M^G that has the following properties:

1. The leaves of T_M^G are the vertices of G.
2. For an internal node h of T_M^G, let $M(h)$ be the set of vertices of G that are leaves of the subtree of T_M^G rooted at h. ($M(h)$ forms a module in G).
3. For each internal node h of T_M^G there is a graph G_h (*representative graph*) with $V(G_h) = \{h_1, h_2, \cdots, h_r\}$, where $h_1, h_2, \cdots, h_r$ are the children of h in T_M^G and for $1 \leq i < j \leq r$, h_i and h_j are adjacent in G_h iff there are vertices $u \in M(h_i)$ and $v \in M(h_j)$ that are adjacent in G.
4. G_h is either a clique, an independent set, or a prime graph and h is labeled *Series* if G_h is clique, *Parallel* if G_h is an independent set, and *Prime* otherwise.

James et al. [17] gave first polynomial time algorithm for finding a modular decomposition which runs in $O(n^4)$ time. Linear time algorithms to find modular decompositions are proposed in [10,24].

A vertex partition (A, B) of a graph G is a *split* if $\tilde{A} = A \cap N(B)$ and $\tilde{B} = B \cap N(A)$ forms a biclique. A split is trivial if $|A|$ or $|B|$ is one. Split decomposition was introduced by Cunningham [11]. Loosely it is the result of a recursive process of decomposing a graph into components based on the splits. Cunningham [11] showed that a graph can be decomposed uniquely into components that are stars, cliques, or prime (i.e., without proper splits). This decomposition is known as the *skeleton*. For details see [12]. A polynomial time algorithm for computing the skeleton of a graph is given in [21].

Theorem 1. *[12](see [6]) Let G be a connected graph. Then the skeleton of G is unique, and the proper splits of G correspond to the special edges of its skeleton and to the proper splits of its complete and star components.*

Organization of the paper: In Section 3 we discuss GI-completeness of prime graph isomorphism. In Section 4 we define a notion of equivalence of parse trees called *structural isomorphism*, and give an algorithm to test if two parse trees are structurally isomorphic. We give an overview of the CHLRR algorithm [6] in Section 5. In Section 6, we present the isomorphism algorithm for prime graphs of clique-width at most three. In Appendix, we show that the CHLRR algorithm can be modified suitably to output structurally isomorphic parse trees for isomorphic graphs.

3   Completeness of Prime Graph Isomorphism

It is known that isomorphism problem for prime graphs is GI-complete [4]. There is an easy polynomial time many-one reduction from GI to prime graph isomorphism[1] described in Lemma 9 of the Appendix. Unfortunately, this reduction does not preserve the clique-width. We also give a clique-width preserving Turing reduction from GI to prime graph isomorphism which we use in our main algorithm. The reduction hinges on the following lemma.

Lemma 1. *[8] G is a graph of clique-width at most k iff each prime graph associated with the modular decomposition of G is of clique-width at most k.*

We next show that if we have an oracle for GI for colored prime graphs of clique-width at most k then there is a GI algorithm for graphs with clique-width at most k.

Theorem 2. *Let $\mathcal{A}'$ be an algorithm that given two colored prime graphs G' and H' of clique-width at most k, decides if $G' \cong H'$ via a color preserving isomorphism. Then there exists an algorithm $\mathcal{A}$ that on input any colored graphs G and H of clique-width at most k decides if $G \cong H$ via a color preserving isomorphism.*

Proof. Let G and H be two colored graphs of clique-width at most k. The algorithm is similar to [13], which proceeds in a bottom up approach in stages

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[1] In fact, it is an AC^0 reduction

starting from the leaves to the root of the modular decomposition trees T_G and T_H of G and H respectively. Each stage corresponds to a level in the modular decomposition. In every level, the algorithm $\mathcal{A}$ maintains a table that stores whether for each pair of nodes x and y in T_G and T_H the subgraphs $G[x]$ and $H[y]$ induced by leaves of subtrees of T_G and T_H rooted at x and y are isomorphic. For the leaves it is trivial to store such information. Let u and v be two internal nodes in the modular decomposition trees of T_G and T_H in the same level. To decide if $G[u]$ and $H[v]$ are isomorphic $\mathcal{A}$ does the following.

If u and v are both *series* nodes then it just checks if the children of u and v can be isomorphically matched. The case for *parallel* node is similar. If u and v are *prime* nodes then the vertices of representative graphs G_u and H_v are colored by their isomorphism type i.e., two internal vertices u_1 and u_2 of the representative graphs will get the same color iff subgraphs induced by leaves of subtrees of T_G (or T_H) rooted at u_1 and u_2 are isomorphic. To test $G[u] \cong H[v]$, $\mathcal{A}$ calls $\mathcal{A}'(\widehat{G}_u, \widehat{H}_v)$, where $\widehat{G}_u$ and $\widehat{H}_v$ are the colored copies of G_u and H_v respectively. At any level if we can not find a pairwise isomorphism matching between the internal nodes in that level of T_G and T_H then $G \cong H$. In this manner we make $O(n^2)$ calls to algorithm $\mathcal{A}'$ at each level. The total runtime of the algorithm is $O(n^3)T(n)$, where $T(n)$ is run time of $\mathcal{A}'$. Note that by Lemma 1 clique-width of G_u and H_v are at most k. $\qquad\square$

4   Testing Isomorphism between Parse Trees

In this section we define a notion of equivalence of parse trees called *structural isomorphism*, and we give an algorithm to test if two given parse trees are equivalent under this notion. As we will see, the graphs generated by equivalent parse trees are always isomorphic. Thus, if we have two equivalent parse trees for the two input graphs, the isomorphism problem indeed admits a polynomial time algorithm. In Section 6, we prove that the CHLRR algorithm can be tweaked slightly to produce structurally isomorphic parse trees for isomorphic graphs with clique-width at most three and thus giving a polynomial-time algorithm for such graphs.

Let G and H be two colored graphs. A bijective map $\pi : V(G) \rightarrow V(H)$ is *color consistent* if for all vertices u and v of G, $color(u) = color(v)$ iff $color(\pi(u)) = color(\pi(v))$. Let $\pi : V(G) \rightarrow V(H)$ be a color consistent mapping, define $\pi/color : color(G) \rightarrow color(H)$ as follows: for all c in $color(G)$, $\pi/color(c) = color(\pi(v))$ where $color(v) = c$. It is not hard to see that the map $\pi/color$ is well defined. Recall that the internal nodes of a parse tree are $\eta_{i,j}$, $\rho_{i \rightarrow j}$ and $\oplus$ operations. The levels of a parse tree correspond to $\oplus$ nodes. Let T_g be a parse tree of G rooted at $\oplus$ node g. Let g_1 be descendant of g which is neither η nor ρ. We say that g_1 is an *immediate significant descendant* of g if there is no other $\oplus$ node in the path from g to g_1. For an immediate significant descendant g_1 of g, we construct a *colored quotient graph* Q_{g_1} that corresponds to graph operations appearing in the path from g to g_1 performed on graph G_{g_1}, where G_{g_1} is graph generated by parse tree T_{g_1}. The vertices of Q_{g_1} are labels of

G_{g_1}. The colors and the edges of Q_{g_1} are determined by the operations on the path from g_1 to g. We start with coloring a vertex a by color a and no edges. If the operation performed is $\eta_{a,b}$ on G_{g_1} then add edges between vertices of color a and color b. If the operation is $\rho_{a \to b}$ on G_{g_1} then recolor the vertices of color a with color b. After taking care of an operation we move to the next operation on the path from g_1 to g until we reach $\oplus$ node g. Notice that if the total number of labels used in a parse tree is k then the size of any colored quotient graph is at most k.

Definition 2. *Let T_g and T_h be two parse trees of G and H rooted at $\oplus$ nodes g and h respectively. We say that T_g and T_h are* structurally isomorphic *via a label map π (denoted $T_g \cong^\pi T_h$)*

1. *If T_g and T_h are single nodes[2] or inductively,*
2. *If T_g and T_h are rooted at g and h having immediate significant descendants $g_1, \cdots, g_r$ and $h_1, \cdots, h_r$, and there is a bijection $\gamma : [r] \to [r]$ and for each i there is a $\pi_i \in \mathsf{ISO}(Q_{g_i}, Q_{h_{\gamma(i)}})$ such that $T_{g_i} \cong^{\pi_i} T_{h_{\gamma(i)}}$ and $\pi_i/color = \pi|_{color(Q_{g_i})}$, where $T_{g_1}, \cdots, T_{g_r}$ and $T_{h_1}, \cdots, T_{h_r}$ are the subtrees rooted at $g_1, \cdots, g_r$ and $h_1, \cdots, h_r$ respectively[3]*

We say that T_g and T_h are structurally isomorphic *if there is a π such that $T_g \cong^\pi T_h$.*

The structural isomorphism is an equivalence relation: reflexive and symmetric properties are immediate from the above definition. The following lemma shows that it is also transitive.

Lemma 2. *Let T_{g_1}, T_{g_2} and T_{g_3} be the parse trees of G_1, G_2 and G_3 respectively such that $T_{g_1} \cong^{\pi_1} T_{g_2}$ and $T_{g_2} \cong^{\pi_2} T_{g_3}$ then $T_{g_1} \cong^{\pi_2 \pi_1} T_{g_3}$.*

Proof. The proof is by induction on the height of the parse trees. The base case trivially satisfies the transitive property. Assume that g_1, g_2 and g_3 are nodes of height $d + 1$. Let g_{1i} be an immediate significant descendant of g_1. Since $T_{g_1} \cong^{\pi_1} T_{g_2}$, there is an immediate significant descendant g_{2j} of g_2 and $\pi_{1i} \in \mathsf{ISO}(Q_{g_{1i}}, Q_{g_{2j}})$ such that $\pi_{1i}/color = \pi|_{color(Q_{g_{1i}})}$ and $T_{g_{1i}} \cong^{\pi_{1i}} T_{g_{2j}}$. Similarly, g_{2j} will be matched to some immediate significant descendant g_{3k} of g_3 via $\pi_{2j} \in \mathsf{ISO}(Q_{g_{2j}}, Q_{g_{3k}})$ such that $\pi_{2j}/color = \pi|_{color(Q_{g_{2j}})}$ and $T_{g_{2j}} \cong^{\pi_{2j}} T_{g_{3k}}$. The nodes g_{1i}, g_{2j} and g_{3k} has height at most d. Therefore, by induction hypothesis $T_{g_{1i}} \cong^{\pi_{2j} \pi_{1i}} T_{g_{3k}}$. By transitivity of isomorphism we can say $\pi_{2j} \pi_{1i} \in \mathsf{ISO}(Q_{g_{1i}}, Q_{g_{3k}})$. To complete the proof we just need to show $\pi_{2j} \pi_{1i}/color = \pi_2 \pi_1|_{color(Q_{g_{1i}})}$. This can be inferred from the following two facts:
1) $\pi_{2j} \pi_{1i}/color = \pi_{2j}/color \; \pi_{1i}/color$
2) $\pi_2 \pi_1|_{color(Q_{g_{1i}})} = \pi_2|_{color(Q_{g_{2j}})} \; \pi_1|_{color(Q_{g_{1i}})}$. $\qquad\qquad\square$

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[2] In this case they are trivially structurally isomorphic via π.

[3] Notice that this definition implies that G_{g_i} and $H_{h_{\gamma(i)}}$ are isomorphic via the label map π_i where G_{g_i} and $H_{h_{\gamma(i)}}$ are graphs generated by the parse trees T_{g_i} and $T_{h_{\gamma(i)}}$ respectively.

Algorithm to Test Structural Isomorphism: Next we describe an algorithm that given two parse trees T_G and T_H tests if they are structurally isomorphic. From the definition if $T_G \cong^\pi T_H$ then we can conclude that G and H are isomorphic. We design a dynamic programming algorithm that basically checks the local conditions 1 and 2 in Definition 2.

The algorithm starts from the leaves of parse trees and proceeds in levels where each level corresponds to $\oplus$ operations of parse trees. Let g and h denotes the $\oplus$ nodes at level l of T_G and T_H respectively. At each level l, for each pair of $\oplus$ nodes $(g, h) \in (T_G, T_H)$, the algorithm computes the set $R_l^{g,h}$ of all bijections $\pi : lab(G_g) \to lab(H_h)$ such that $G_g \cong_f^\pi H_h$ for some f, and stores in a table indexed by (l, g, h), where G_g and H_h are graphs generated by sub parse trees T_g and T_h rooted at g and h respectively. To compute $R_l^{g,h}$, the algorithm uses the already computed information $R_{l+1}^{g_i,h_j}$ where g_i and h_j are immediate significant descendants of g and h.

The base case correspond to finding $R_l^{g,h}$ for all pairs (g, h) such that g and h are leaves. Since in this case G_g and H_h are just single vertices, it is easy to find $R_l^{g,h}$. For the inductive step let $g_1, \cdots, g_r$ and $h_1, \cdots, h_{r'}$ be the immediate significant descendants of g and h respectively. If $r \neq r'$ then $R_l^{g,h} = \emptyset$. Otherwise we compute $R_l^{g,h}$ for each pair (g, h) at level l with help of the already computed information up to level $l + 1$ as follows.

For each $\pi : lab(G_g) \to lab(H_h)$ and pick g_1 and try to find a h_{i_1} such that $T_{g_1} \cong^{\pi_1} T_{h_{i_1}}$ for some $\pi_1 \in \mathsf{ISO}(Q_{g_1}, Q_{h_{i_1}}) \cap R_{l+1}^{g_1,h_{i_1}}$ such that $\pi_1/color = \pi|_{color(Q_{g_1})}$. We do this process to pair g_2 with some unmatched h_{i_2}. Continue in this way until all immediate significant descendants are matched. By Lemma 3, we know that this greedy matching satisfies the conditions of Definition 2. If all the immediate significant descendants are matched we add π to $R_l^{g,h}$. It is easy to see that if $R_l^{g,h} \neq \emptyset$ then the subgraphs $G_g \cong_f^\pi H_h$ for $\pi \in R_l^{g,h}$. From the definition of structurally isomorphic parse trees it is clear that if $R_0^{g,h} \neq \emptyset$ then $G \cong H$. The algorithm is polynomial time as the number of choices for π and π_1 is at most $k!$ which is a constant, where $|lab(G)| = k$.

Note that for colored graphs, by ensuring that we only match vertices of same color in the base case, the whole algorithm can be made to work for colored graphs. In Lemma 2 we prove that structural isomorphism satisfies transitivity. In fact, structural isomorphism satisfies a stronger notion of transitivity as stated in the following lemma.

Lemma 3. *Let T_g and T_h be two parse trees of graphs G and H. Let g_1 and g_2 be two immediate significant descendants of g, and h_1 and h_2 be two immediate significant descendants of h. Suppose for $i = 1, 2$, $T_{g_i} \cong^{\pi_i} T_{h_i}$ for some $\pi_i \in \mathsf{ISO}(Q_{g_i}, Q_{h_i})$ with $\pi_i/color = \pi|_{color(Q_{g_i})}$. Also assume that $T_{g_1} \cong^{\pi_3} T_{h_2}$ where $\pi_3 \in \mathsf{ISO}(Q_{g_1}, Q_{h_2})$ and $\pi_3/color = \pi|_{color(Q_{g_1})}$. Then, $T_{g_2} \cong^{\pi_1 \pi_3^{-1} \pi_2} T_{h_1}$ where $\pi_1 \pi_3^{-1} \pi_2 \in \mathsf{ISO}(Q_{g_2}, Q_{h_1})$ and $\pi_1 \pi_3^{-1} \pi_2/color = \pi|_{color(Q_{g_2})}$.*

Proof. By Lemma 2, $T_{g_2} \cong^{\pi_1 \pi_3^{-1} \pi_2} T_{h_1}$. The rest of the proof is similar to the proof of the inductive case of Lemma 2. □

5 Overview of the CHLRR Algorithm

Corneil et al. [6] gave the first polynomial time algorithm (the *CHLRR algorithm*), to recognize graphs of clique-width at most three. We give a brief description of their algorithm in this section. We mention that our description of this fairly involved algorithm is far from being complete. The reader is encouraged to see [6] for details. By Lemma 1 we assume that the input graph G is prime.

To test whether clique-width of prime graph G is at most three the algorithm starts by constructing a set of bilabelings and trilabelings of G. In general the number of bilabelings and trilabelings are exponential, but it was shown (Lemma 8 and 9 in [6]) that it is enough to consider the following linear size subset denoted by $LabG$.

1. For each vertex v in $V(G)$
   [B_1] Generate the bilabeling[4]$\{v\}$ and add it to $LabG$.
   [B_2] Generate the bilabeling $\{x \in N(v) \mid N[x] \subseteq N[v]\}$ and add it to $LabG$.
2. Compute the skeleton of G search this skeleton for the special edges, clique and star components.
   [T_1] For each special edge s (corresponds to a proper split), generate the trilabeling $\tilde{X}, \tilde{Y}, V(G) \setminus (\tilde{X} \cup \tilde{Y})$ where (X,Y) is the split defined by s and add it to $LabG$.
   [B_3] For all clique components C, generate the bilabeling C and add it to $LabG$.
   [B_4] For all star components S, generate the bilabeling $\{c\}$, where c is the special center of S, and add it to $LabG$.

Lemma 4. *[6] Let G be a prime graph. Clique-width of G is at most three if and only if at least one of the bilabelings or trilabelings in LabG has clique-width at most three.*

By Lemma 4 the problem of testing whether G is of clique-width at most three is reduced to checking one of labeled graph in $LabG$ is of clique-width at most three. To test if a labeled graph A taken from $LabG$ is of clique-width at most three, the algorithm follows a top down approach by iterating over all possible last operations that arise in the parse tree representation of G. For example, for each vertex x in G the algorithm checks whether the last operation must have joined x with its neighborhood. In this case the problem of testing whether G can be constructed using at most three labels is reduced to test whether $G \setminus \{x\}$ can be constructed using at most three lables. Once the last operations are fixed the original graph decomposes into smaller components, which can be further decomposed recursively.

For each A in $LabG$, depending on whether it is bilabeled or trilabeled the algorithm makes different tests on A to determine whether A is of clique-width

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[4] *bilabeling* of a set $X \subseteq V$ indicates that all the vertices in X are labeled with one label and $V \setminus X$ is labeled with another label.

at most three. Based on the test results the algorithm either concludes clique-width of A is more than three or returns top operations of the parse tree for A along with some connected components of A which are further decomposed recursively.

If A in $LabG$ is connected, trilabeled (with labels l_1, l_2, l_3) and l-prime then by the construction of $LabG$, A corresponds to a split (possibly trivial). If A has a proper split then there exists $a \neq b$ in $\{l_1, l_2, l_3\}$ such that A will be disconnected with the removal of edges E_{ab}. This gives a decomposition with top operations $\eta_{a,b}$ followed by a $\oplus$ node whose children are connected components of $A \setminus E_{ab}$. If A has a universal vertex v (trivial split) labeled a in A then by removing edges E_{ab} and E_{ac} we get a decomposition with top operations $\eta_{a,b}$ and $\eta_{a,c}$ followed by a $\oplus$ operation with children connected components of $A \setminus (E_{ab} \cup E_{ac})$.

To describe the bilabeled case we use V_i to denote the set of vertices of A with label i. If A in $LabG$ is connected, bilabeled (with labels l_1, l_2) and l-prime, then the last operation is neither η_{l_1,l_2} (otherwise A will have a l-module) nor $\oplus$ (A is connected). So the last operation of the decomposition must be a relabeling followed by a join operation i.e., we have to introduce a third label set V_{l_3} such that all the edges are present between the two of three labeled sets.

After introducing third label if there is only one join to undo, then we have a unique way to decompose the graph into smaller components. If there are more than one possible join to be removed, then it is enough to consider one of them and proceed (see Section 5.2 in [6]). There are four ways to introduce the third label to decompose the graph, but they might correspond to overlapping cases. To overcome this the algorithm first checks whether A belongs to any of three simpler cases described below.

PC1: A has a universal vertex x of label $l \in \{l_1, l_2\}$. In this case relabel vertex x with l_3 and remove the edges $E_{l_3 l_2}$, and $E_{l_3 l_1}$ to decompose A. This gives a decomposition with $\rho_{l_3 \to l}$, η_{l_3,l_2}, η_{l_3,l_1} followed by $\oplus$ operation with children x and $A \setminus \{x\}$.

PC2: A has a vertex x of label $l \in \{l_1, l_2\}$ that is universal to all vertices of label $l' \in \{l_1, l_2\}$, but is not adjacent to all vertices with the other label, say $\bar{l}'$. In this case relabel vertex x with l_3 and remove the edges $E_{l_3 l'}$. This gives a decomposition with $\rho_{l_3 \to l}$, $\eta_{l_3,l'}$ above a $\oplus$ operation with children x and $A \setminus \{x\}$.

PC3: A has two vertices x and y of label l, where y is universal to everything other than x, and x is universal to all vertices of label l other than y, and non-adjacent to all vertices with the other label $\bar{l}$. In this case the algorithm relabels vertices x and y with l_3, and by removing edges $E_{l_3 l}$ disconnects the graph A, with two connected components x and $A \setminus \{x\}$. Now in graph $A \setminus \{x\}$ again remove the edges $E_{l_3 \bar{l}}$ to decompose the graph into two parts y and $A \setminus \{x, y\}$.

If A does not belongs to any of above three simpler cases then there are four different ways to introduce the third label set to decompose the graph as described below.

Let $\mathcal{E}$ be the set of all connected, bilabeled, l-prime graphs with clique-width at most three and not belonging to above three simpler cases. For $l \in \{1, 2\}$ we define the following four subsets of $\mathcal{E}$.

1. $\mathcal{U}_l$: $V_l^a \neq \emptyset$ and removing the edges between the V_l^a and $V_{\bar{l}}$ disconnects the graph.

2. $\overline{\mathcal{D}}_l$: $\overline{V}_l$ is not connected and removing the edges between the coconnected components of $\overline{V}_l$ disconnects the graph.

In these four cases the algorithm introduces a new label l_3 and removes the edges E_{ll_3}, $l \in \{l_1, l_2\}$ to disconnect A. This gives a decomposition with $\rho_{l_3 \to l}$ and η_{l,l_3} followed by $\oplus$ operation with children that are the connected components of $A \setminus E_{ll_3}$. For more details about decomposition process when A is in $\mathcal{U}_l$ or $\overline{\mathcal{D}}_l$, $l \in \{1, 2\}$ the reader is encouraged to see Section 5.2 in [6].

The following Lemma shows that there is no other possible way of decomposing a clique-width at most three graphs apart from the cases described above.

Lemma 5. *[6] $\mathcal{E} = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \overline{\mathcal{D}}_1 \cup \overline{\mathcal{D}}_2$, and this union is disjoint.*

In summary, for any labeled graph A in *LabG* the CHLRR algorithm tests whether A belongs to any of the above described cases, if it is then it outputs suitable top operations and connected components. The algorithm continues the above process repeatedly on each connected component of A until it either returns a parse tree or concludes clique-width of A is more than three.

6 Isomorphism Algorithm for Prime Graphs of Clique-width at most Three

In Section 4 we described algorithm to test structural isomporphism between two parse trees. In this Section we show that given two isomorphic prime graphs G and H of clique-width at most three, the CHLRR algorithm can be slightly modified to get structurally isomorphic parse trees. We have used four labels in order to preserve structural isomorphism in the modified algorithm. The modified algorithm is presented in Appendix. Recall that the first step of the CHLRR algorithm is to construct a set *LabG* of bilabelings and trilabelings of G as described in Section 5.

Definition 3. *We say that LabG is equivalent to LabH denoted as $LabG \equiv LabH$ if there is a bijection $g : LabG \to LabH$ such that for all $A \in LabG$, there is an isomorphism $f : V(A) \to V(g(A))$ and a bijection $\pi : lab(A) \to lab(g(A))$ such that $A \cong_f^\pi g(A)$.*

Lemma 6. *$LabG \equiv LabH$ iff $G \cong_f H$.*

Proof. The proof follows from the construction of sets *LabG* and *LabH* from input prime graphs G and H and it is presented in Appendix. □

Lemma 7. *Let $A \in LabG$ and $B \in LabH$. If $A \cong_f^\pi B$ for some f and π then parse trees generated from Decompose function (Algorithm 2) for input graphs A and B are structurally isomorphic. More specifically, $Decompose(A) \cong_f^\pi Decompose(B)$.*

Proof. Follows from Lemma 11 and Lemma 12 described in Appendix. The major modifications are done in PC2 case, where we have used four labels in order to preserve structural isomorphism between parse trees. □

Isomorphism Algorithm

For two input prime graphs G and H the algorithm works as follows. Using modified CHLRR algorithm, first a parse tree T_G of clique-width at most three is computed for G. The parse tree T_G of G is not canonical but from Lemma 6 and 7, we know that if $G \cong H$ then there exists parse tree T_H of H, structurally isomorphic to T_G. Therefore we compute parse tree of clique-width at most three for each labeled graph in $LabH$. For each such parse tree T_H, the algorithm uses the structural isomorphic algorithm described in Section 4 to test the structural isomorphism between parse trees T_G and T_H. If $T_G \cong T_H$ for some T_H, then we conclude that $G \cong H$. If there is no parse tree of H which is structurally isomorphic to T_G then G and H can not be isomorphic.

Computing a parse tree T_G of G takes $O(n^2m)$ time. As there are $O(n)$ many labeled graphs in $LabH$, computing all possible parse trees for labeled graphs in $LabH$ takes $O(n^3m)$ time. Testing structural isomorphism between two parse trees need $O(n^3)$ time. Therefore the running time to check isomorphism between two prime graphs G and H of clique-width at most three is $O(n^3m)$. □

The correctness of the algorithm follows from Lemma 8 and Theorem 3. Lemma 8 shows that if $G \cong H$ then we can always find two structurally isomorphic parse trees T_G and T_H using the modified CHLRR algorithm.

Lemma 8. *Let G and H be prime graphs with clique-width at most three. If $G \cong_f H$ then for every T_G in parseG there is a T_H in parseH such that T_G is structurally isomorphic to T_H where parseG and parseH are the set of parse trees generated by Algorithm 1 on input LabG and LabH respectively.*

Proof. If $G \cong_f H$ then from Lemma 6 we have $LabG \equiv LabH$ i.e., for every A in $LabG$ there is a $B = g(A)$ in $LabH$ such that $A \cong_f^\pi B$ for some f and π. On input such A and B to Lemma 7 we get two parse trees T_A and T_B which are structurally isomorphic. □

Theorem 3. *Let G and H be graphs with clique-width at most three. Then there exists a polynomial time algorithm to check whether $G \cong H$.*

Proof. The proof follows from the prime graph isomorphism of graphs with clique-width at most three described in Lemma 8 and Theorem 2. □

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Appendix

7 Graph Isomorphism Completeness for Prime Graphs

For each vertex $v \in V(G)$, the polynomial-time many-one reduction adds a new vertex v' and adds an edge between v and v' to get a new graph $\widehat{G}$. After the addition of vertices and edges to the graph it is easy to see that each old vertex in the graph is adjacent to exactly one vertex of degree one. It is not hard to see that if M is a non-trivial module in a graph then no vertex in M is adjacent to a vertex of degree one. Thus, we can conclude that $\widehat{G}$ is prime graph.

Lemma 9. *Given two connected graphs G_1 and G_2, $G_1 \cong G_2$ iff $\widehat{G}_1 \cong \widehat{G}_2$.*

Proof. Let $\widehat{G}_1$ and $\widehat{G}_2$ are graphs obtained after adding new vertices to G_1 and G_2 respectively. If $G_1 \cong_f G_2$ then we can find an isomorphism between $\widehat{G}_1$ and $\widehat{G}_2$ by extending f to newly added vertices such that for every new vertex $y \in \widehat{G}_1$ having neighbor x, f maps y to z, where z is the newly added neighbor of $f(x)$ in $\widehat{G}_2$. For the other direction when $\widehat{G}_1 \cong_f \widehat{G}_2$, as there are no old vertices of degree one in $\widehat{G}_1$ and $\widehat{G}_2$ any isomorphism f from $\widehat{G}_1$ to $\widehat{G}_2$ must map the old vertices of $\widehat{G}_1$ to the old vertices of $\widehat{G}_2$. The restriction of f to the old vertices of $\widehat{G}_1$ and $\widehat{G}_2$ is an isomorphism from G_1 to G_2. $\qquad \square$

Lemma 10. *$LabG \equiv LabH$ iff $G \cong_f H$.*

Proof. It is easy to see that if $LabG \equiv LabH$ then $G \cong_f H$ from the definition. For the other direction, given two graphs G and H isomorphic via f, we need to prove that there is a bijection $g : LabG \to LabH$ such that for all $A \in LabG$, there is an isomorphism $f : V(A) \to V(g(A))$ and a bijection $\pi : lab(A) \to lab(g(A))$ such that $A \cong_f^\pi g(A)$.

  The proof is divided into five cases based on how bilabelings and trilabelings are generated by CHLRR algorithm described in Section 5.

1. Let $A \in LabG$ be generated at B_1 in CHLRR algorithm. Therefore, A has bilabeling $\{v\}$. Since $G \cong_f H$, there is a graph $B \in LabH$ which has bilabeling $\{f(v)\}$. Define $g(A) = B$ and a bijection $\pi : lab(A) \to lab(B)$ such that if $lab(v) = i$ then $\pi(i) = lab(f(v))$ so that $A \cong_f^\pi B$.
2. Let $A \in LabG$ be generated at B_2. Thus, A has bilabeling $P = \{x \in N(v)|N[x] \subseteq N[v]\}$. As $G \cong_f H$, there is a graph $B \in LabH$ with bilabeling $f(P) = \{f(x) \in N(f(v))|N[f(x)] \subseteq N[f(v)]\}$. Define $g(A) = B$ and a bijection $\pi : lab(A) \to lab(B)$ such that if $lab(P) = i$ then $\pi(i) = lab(f(P))$ so that $A \cong_f^\pi B$.
3. Let $A \in LabG$ be generated at T_1 for a special edge s in the skeleton of G with trilabeling $\tilde{X}$, $\tilde{Y}$, $V(G) \setminus (\tilde{X} \cup \tilde{Y})$. As $G \cong_f H$ and the skeleton of graph is unique (from Theorem 1), we can find a $B \in LabH$ which is generated for the special edge $f(s)$ in skeleton of H which corresponds to trilabeling $f(\tilde{X})$, $f(\tilde{Y})$, $V(H) \setminus (f(\tilde{X} \cup \tilde{Y}))$. Define $g(A) = B$ and a bijection $\pi : lab(A) \to lab(B)$ such that if $lab(\tilde{X}) = i_1$ then $\pi(i_1) = lab(f(\tilde{X}))$, if

$lab(\tilde{Y}) = i_2$ then $\pi(i_2) = lab(f(\tilde{Y}))$, and if $lab(V(G) \setminus (\tilde{X} \cup \tilde{Y})) = i_3$ then $\pi(i_3) = lab(V(H) \setminus (f(\tilde{X} \cup \tilde{Y})))$ so that $A \cong_f^\pi B$.

4. Let $A \in LabG$ be generated at B_3 for a clique component C with bilabeling C. As $G \cong_f H$, there is a $B \in LabH$ which is generated for a clique component $f(C)$ with bilabeling $f(C)$. Define $g(A) = B$ and a bijection $\pi : lab(A) \to lab(B)$ such that if $lab(C) = i$ then $\pi(i) = lab(f(C))$ so that $A \cong_f^\pi B$.

5. Let $A \in LabG$ be generated at B_4 for a star component S with bilabeling $\{c\}$, where c is a special center of S. As $G \cong_f H$, there is a graph $B \in LabH$ which is generated for a star component $f(S)$ with bilabeling $f(c)$, where $f(c)$ is a special center of $f(S)$. Define $g(A) = B$ and a bijection $\pi : lab(A) \to lab(B)$ such that if $lab(c) = i$ then $\pi(i) = lab(f(c))$ so that $A \cong_f^\pi B$. $\qquad \square$

8 Generating Structurally Isomorphic Parse Trees

In this section we prove that the modified CHLRR algorithm generates structurally isomorphic parse trees on two isomorphic input graphs. To prove that we also show that the supporting subroutines do the same.

Algorithm 1: Finding parse trees for labeled graphs of clique-width at most three in $LabG$

    Input: $LabG$ a set of bilabelings and trilabelings of G
    Output: $parseG = \{T_A \mid T_A$ is a parse tree of graph A in LabG of clique-width at most three $\}$

1 **begin**
2    $parseG := \emptyset$
3   **for all** $A \in LabG$ **do**
4       $A.parse\text{-}tree := null$
5       $A.parse\text{-}tree = Decompose(A)$
6   **if** $A.parse\text{-}tree \neq null$ **then**
7   Add $A.parse\text{-}tree$ to $parseG$

8   return $(parseG)$

The function $Decompose(P)$ in Algorithm 1 finds parse tree of P if $cwd(P) \leq 3$ and it is described in following Section and Appendix.

8.1 Decomposing Trilabeled Graphs

The function $Decompose\text{-}leaf\text{-}TI$ (Algorithm 3) decomposes trilabeled graph from $LabG$. It can be check that this function is always called with inputs coming from $LabG$. In other words it is only called in the first level of the recursion.

Lemma 11. *Let A in $LabG$ and B in $LabH$ be trilabeled and l-prime connected graphs. If $A \cong_f^\pi B$ for some f and π then Algorithm 3 generates top operations of parse trees for A and B such that $\pi \in \mathsf{ISO}(Q_a, Q_b)$ with $A_a \cong_f^\pi B_b$, where A_a and B_b are the graphs described in Algorithm 3.*

Proof. Let A and B are trilabeled with l_1, l_2, l_3 and l_1', l_2', l_3' respectively. If A has a trivial split (see Figure 1) then it has a universal vertex x of some label l_1. Then

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Algorithm 2: Function Decompose [6]

Input: A bi or trilabled l-prime connected graph P
Output: A parse tree of P or null parse tree if $cwd(P) > 3$

1 **begin**
2     $parse\text{-}tree :=$ a trivial parse tree with P as the unique leaf
      /* $parse\text{-}tree$ may contain connected components as leafs but as the algorithm proceeds
      this components will be decomposed to finally obtain the parse tree   */
3     $Leaves := \{P\}$ /* $Leaves$ contains pointer to P */
4   **while** $Leaves \neq \emptyset$ **do**
5   flag := true, tree := null
6   Extract Γ from $Leaves$
7   **if** Γ *has no more than three vertices* **then**
8   Find a canonical parse tree, *tree*
9   Replace Γ by *tree* in *parse-tree*
10   **if** Γ *is trilabled* **then**
11   [flag, tree] = Decompose-leaf-TI(Γ)
12   Add the leafs of *tree* to *Leaves*
13   Replace Γ by *tree* in *parse-tree*
14   **else**
15   [flag, tree] = Decompose-leaf-BI(Γ)
16   Add the leafs of *tree* to *Leaves*
17   Replace Γ by *tree* in *parse-tree*
18   **if** *flag is false* **then**
19   *parse-tree* := null
20   return (*parse-tree*)

21   return (*parse-tree*)

---

the algorithm removes the edges $E_{l_1 l_2}$, $E_{l_1 l_3}$ from A and gives a decomposition with top operations η_{l_1,l_2} and η_{l_1,l_3} above a $\oplus$ operation whose children are x and connected components $A_{a_1}, \cdots, A_{a_k}$ ($A_a = x \oplus A_{a_1} \oplus \cdots \oplus A_{a_k}$). If $A \cong^\pi_f B$, then there is a universal vertex y in B of label l'_1 such that $f(x) = y$ and $\pi(l_1) = l'_1$. To decompose B, the algorithm removes the edges $E_{l'_1 l'_2}$, $E_{l'_1 l'_3}$ from B to get the decomposition with top operations $\eta_{l'_1 l'_2}$ and $\eta_{l'_1 l'_3}$ above a $\oplus$ operation whose children are y and connected components $B_{b_1}, \cdots, B_{b_k}$ ($B_b = y \oplus B_{b_1} \oplus \cdots \oplus B_{b_k}$). In fact $B_{b_1}, \cdots, B_{b_k}$ are images of $A_{a_1}, \cdots, A_{a_k}$ under f in some order. The quotient graphs Q_a and Q_b have three vertices corresponding to top two consecutive η operations. If $A \cong^\pi_f B$ the quotient graphs are isomorphic via π and $A_a \cong^\pi_f B_b$.

If A corresponds to a nontrivial split (see Figure 2) then there are two labels l_1, l_2 such that $E_{l_1 l_2}$ is complete. We get a decomposition with η_{l_1,l_2} operation above a $\oplus$ operation whose children are connected components $A_{a_1}, \cdots, A_{a_k}$ ($A_a = A_{a_1} \oplus \cdots \oplus A_{a_k}$) of A after the $E_{l_1 l_2}$ edges are removed. If $A \cong^\pi_f B$, then there exists a nontrivial split in B and two labels l'_1, l'_2 such that $E_{l'_1 l'_2}$ is complete and $\pi\{l_1, l_2\} = \{l'_1, l'_2\}$, $\pi(l_3) = l'_3$. To decompose B, the algorithm removes the edges $E_{l'_1 l'_2}$, to get the decomposition with top operations $\eta_{l'_1,l'_2}$ above a $\oplus$ operation whose children are connected components $B_{b_1}, \cdots, B_{b_k}$ obtained from B after $E_{l'_1 l'_2}$ edges are removed. The quotient graphs Q_a and Q_b build from the top operations are isomorphic via π and $A_a \cong^\pi_f B_b$. □

Algorithm 3: Function Decompose-leaf-TI [6]

Input: A trilabeled, l-prime and connected graph G
Output: true with top operations of parse tree or false if $cwd(G) > 3$

1 begin

2 $\quad$ tree := null

3 $\quad$ **if** G *has a universal vertex x of label l_1* **then**

4 $\quad\quad$ Let $G_g = x \oplus_{i=1}^k G_{g_i}$, where x, G_{g_i}'s are connected components of
$\quad\quad$ $G \setminus \{E_{l_1 l_2}, E_{l_1 l_3}\}$ $\qquad\qquad\qquad\qquad$ /* l_1, l_2, l_3 are labels of G */

5 $\quad\quad$ $tree = \eta_{l_1, l_2} \eta_{l_1, l_3} (x \oplus_{i=1}^k G_{g_i})$

6 $\quad\quad$ return (true, tree)

7 $\quad$ **if** G *has two labels l_1, l_2 such that $E_{l_1 l_2}$ is complete* **then**

8 $\quad\quad$ Let $G_g = \oplus_{i=1}^k G_{g_i}$, where G_{g_i}'s are connected components of $G \setminus \{E_{l_1 l_2}\}$

9 $\quad\quad$ $tree = \eta_{l_1, l_2} (\oplus_{i=1}^k G_{g_i})$

10 $\quad\quad$ return (true, tree)

11 $\quad$ return (false, tree) (i.e., $cwd(G) > 3$)

Algorithm 4: Function *Decompose-leaf-BI* (cf., [6])

Input: A bilabeled, l-prime and connected graph G
Output: true with top operations of parse tree or false if $cwd(G) > 3$

1 begin

2 $\quad$ tree := null

3 $\quad$ **if** $G \in PC1$ **then**

4 $\quad\quad$ **if** G *has a universal vertex (say x)* **then**

5 $\quad\quad\quad$ Let $G_g = x \oplus_{i=1}^k G_{g_i}$, where x, G_{g_i}'s are connected components of
$\quad\quad\quad$ $G \setminus \{E_{l_3 l_2}, E_{l_3 l_1}\}$ $\qquad\qquad\qquad\qquad$ /* l_1, l_2 are labels of G */

6 $\quad\quad\quad$ $tree = \rho_{l_3 \to l} \eta_{l_3, l_2} \eta_{l_3, l_1} (x \oplus_{i=1}^k G_{g_i})$

7 $\quad\quad\quad$ return (true, tree)

8 $\quad$ **if** $G \in PC2$ **then**

9 $\quad\quad$ Compute a set S number of vertices in G which are universal to one label class but
$\quad\quad$ not adjacent to other label class

10 $\quad\quad$ **if** $|S|$ *equal to 1(say x)* **then**

11 $\quad\quad\quad$ Let $G_g = x \oplus_{i=1}^k G_{g_i}$, where x, G_{g_i}'s are connected components of $G \setminus \{E_{l_3 l'}\}$

12 $\quad\quad\quad$ $tree = \rho_{l_3 \to l} \eta_{l_3, l'} (x \oplus_{i=1}^k G_{g_i})$

13 $\quad\quad\quad$ return (true, tree)

14 $\quad\quad$ **if** $|S|$ *equal to 2 (say x_1 and x_2)* **then**

15 $\quad\quad\quad$ Let $G_g = x_1 \oplus x_2 \oplus_{i=1}^k G_{g_i}$, where x_1, x_2, G_{g_i}'s are connected components of
$\quad\quad\quad$ $G \setminus \{E_{l_3 l'}, E_{l_4, \overline{l'}}\}$

16 $\quad\quad\quad$ $tree = \rho_{l_4 \to \overline{l'}} \eta_{l_4, \overline{l'}} \rho_{l_3 \to l'} \eta_{l_3, l'} (x_1 \oplus x_2 \oplus_{i=1}^k G_{g_i})$

17 $\quad\quad\quad$ return (true, tree)

18 $\quad$ **if** $G \in PC3$ **then**

19 $\quad\quad$ Let $G_g = x \oplus y \oplus_{i=1}^k G_{g_i}$, where x, y, G_{g_i}'s are connected components of
$\quad\quad$ $G \setminus \{E_{l_3 l}, E_{l_4, \overline{l}}\}$

20 $\quad\quad$ $tree = \rho_{l_3 \to l} \eta_{l_3, l} (x \oplus \eta_{l_3, \overline{l}} (y \oplus_{i=1}^k G_{g_i}))$

21 $\quad\quad$ return (true, tree)

22 $\quad$ Compute the coconnected components of V_{l_1} and V_{l_2} and test membership of G in
$\quad$ $\mathcal{U}_{l_1}, \mathcal{U}_{l_2}, \overline{\mathcal{D}}_{l_1}$ and $\overline{\mathcal{D}}_{l_2}$

23 $\quad$ **if** $G \in \mathcal{U}_{l_1}$ **then** return(Decompose-leaf-$\mathcal{U}_{l_1}(G)$)

24 $\quad$ **if** $G \in \mathcal{U}_{l_2}$ **then** return(Decompose-leaf-$\mathcal{U}_{l_2}(G)$)

25 $\quad$ **if** $G \in \overline{\mathcal{D}}_{l_1}$ **then** return(Decompose-leaf-$\overline{\mathcal{D}}_{l_1}(G)$)

26 $\quad$ **if** $G \in \overline{\mathcal{D}}_{l_2}$ **then** return(Decompose-leaf-$\overline{\mathcal{D}}_{l_2}(G)$)

27 $\quad$ **if** $G \notin \mathcal{U}_{l_1}, \mathcal{U}_{l_2}, \overline{\mathcal{D}}_{l_1}$ and $\overline{\mathcal{D}}_{l_2}$ **then** return(false,tree) (i.e., $cwd(G) > 3$)
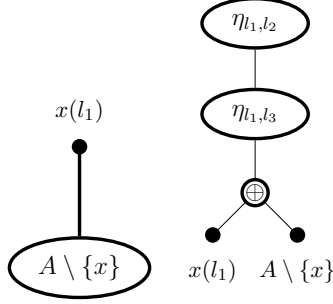

Fig. 1 Trivial split: x is a universal vertex of label l_1 in a trilabeled graph A. We use the bold edge between two sets of vertices to indicate that all edges are present between two vertex sets.
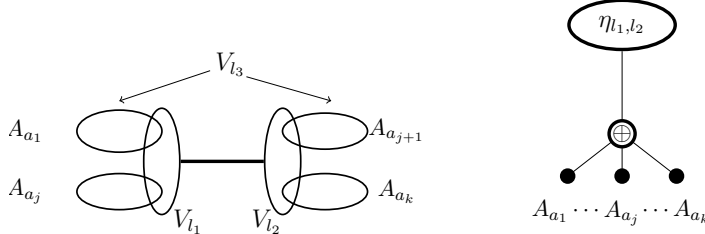

Fig. 2 Nontrivial split: V_i represents set of vertices in A that have label i.

8.2 Decomposing Bilabeled Graphs

Our modification to the CHLRR algorithm is in *Decompose-leaf-BI* (Algorithm 4), where we use four labels instead of three to find structural isomorphic parse trees. If G is a bilabeled, l-prime and connected graph of clique-width at most three, then either $G \in PCi$ where $i \in \{1, 2, 3\}$ or $G \in \mathcal{U}_i, \overline{\mathcal{D}}_i$ where $i \in \{1, 2\}$ (See Proposition 29 in [6]). From here on wards we assume that G and H are bilabeled with l_1, l_2 and l'_1, l'_2 respectively.

Lemma 12. *Let G and H be bilabeled, l-prime and connected graphs. If $G \cong^\pi_f H$ for some f and π then Algorithm 4 generates top operations of parse trees G and H such that there is a $\pi_i \in \mathsf{ISO}(Q_g, Q_h)$ with $G_g \cong^{\pi_i}_f H_h$ and $\pi_i/color = \pi|_{color(Q_g)}$, where G_g and H_h are the graphs described in Algorithm 4.*

Proof. There are three simple cases that can be handled easily. These simple cases denoted as PC1, PC2 and PC3. The other cases $\mathcal{U}_{l_1}$ ($\mathcal{U}_{l_2}$) and $\overline{\mathcal{D}}_{l_1}$ ($\overline{\mathcal{D}}_{l_2}$) are described in Algorithms 5 and 6 in Appendix.

PC1: If $G \in PC1$ then G has a universal vertex of label $l \in \{l_1, l_2\}$ (see Figure 3). Note that in this case G can not have more than two universal vertices of same label, otherwise those universal vertices form an l-module.

To decompose G the algorithm relabels vertex x with l_3 and removes the edges $E_{l_3 l_2}$ and $E_{l_3 l_1}$. Then we get the decomposition with $\rho_{l_3 \to l}$, η_{l_3, l_2}, η_{l_3, l_1} above a $\oplus$ operation with children x and connected components $G_{g_1}, \cdots, G_{g_k}$ ($G_g = x \oplus G_{g_1} \oplus \cdots \oplus G_{g_k}$). If $G \cong_f^\pi H$ then algorithm finds the unique universal vertex y in H of label $l' \in \{l'_1, l'_2\}$ such that $f(x) = y$ and $\pi(l) = l'$. To decompose H the algorithm relabels the vertex y with l'_3 and removes the edges $E_{l'_3 l'_2}$ and $E_{l'_3 l'_1}$ to get the decomposition with $\rho_{l'_3 \to l'}$, $\eta_{l'_3, l'_2}$, $\eta_{l'_3, l'_1}$ above a $\oplus$ operation with children y and connected components $H_{h_1}, \cdots, H_{h_k}$ (these are images of $G_{g_1}, \cdots, G_{g_k}$ under f in some order). The quotient graphs Q_g and Q_h build from the top operations are isomorphic via π_i, where $\pi_i(l_3) = l'_3$ and $\pi_i(l) = \pi(l)$ if $l \in \{l_1, l_2\}$. It is clear that, $G_g \cong_f^{\pi_i} H_h$ and $\pi_i/color = \pi|_{color(Q_g)}$.
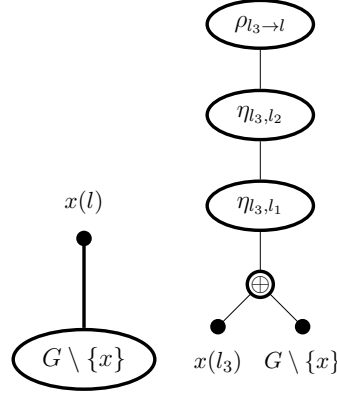

Fig. 3 PC1: Decomposition of a bilabeled graph G with a universal vertex x.

Suppose G has two universal vertices x_1 and x_2 of label l_1 and l_2 respectively. In this case we apply above procedure consecutively two times first taking x_1 as a universal vertex in graph G, second taking x_2 as a universal vertex in graph $G \setminus \{x_1\}$. Note that the order in which we consider x_1 and x_2 does not effect the structure of the parse tree.

PC2: If $G \in PC2$ then G can have one or two vertices of different labels which are universal to vertices of one label class but not to other label class. Let l_1 and l_2 be the labels of G. In this case the algorithm finds the decomposition of G described as follows:

Case-1: Suppose G has a single vertex x of label l (see Figure 4a) that is universal to all vertices of label $l' \in \{l_1, l_2\}$, but not adjacent to all vertices of label $\bar{l}' \in \{l_1, l_2\} \setminus l'$. To decompose G, the algorithm relabels x with a label $l_3 \notin \{l_1, l_2\}$ and removes the edges $E_{l_3 l'}$, which gives the decomposition with

top operations $\rho_{l_3 \to l}$, $\eta_{l_3,l'}$ above a $\oplus$ operation with children x and connected components $G_{g_1}, \cdots, G_{g_k}$ ($G_g = x \oplus G_{g_1} \oplus \cdots \oplus G_{g_k}$). If $G \cong_f^\pi H$, the algorithm finds a vertex y in H of label m which is universal to all vertices of label m' but not adjacent to all vertices of label $\bar{m}'$ such that $f(x) = y$ and $\pi(l) = m$. To decompose H the algorithm relabels y with a label $l_3' \notin \{l_1', l_2'\}$ and removes the edges $E_{l_3' m}$, which gives the decomposition with top operations $\rho_{l_3' m}$, $\eta_{l_3', m'}$ above a $\oplus$ operation whose children are y and the connected components $H_{h_1}, \cdots, H_{h_k}$ (these are images of $G_{g_1}, \cdots, G_{g_k}$ under f in some order). The quotient graphs Q_g and Q_h build from top operations are isomorphic via π_i, where $\pi_i(l) = m$, $\pi(\bar{l}) = \bar{m}$ and $\pi_i(l_3) = l_3'$. Moreover, $G_g \cong_f^{\pi_i} H_h$ and $\pi_i/color = \pi|_{color(Q_g)}$.
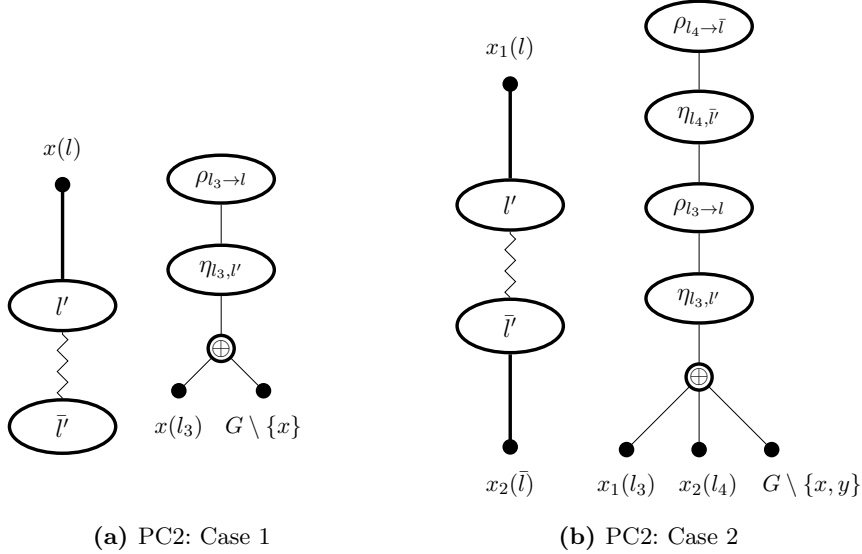

(a) PC2: Case 1   **(b)** PC2: Case 2

Fig. 4 Decomposing a bilabeled graph G, having one or two vertices of different labels which are universal to vertices of one label class but not to other label class. We use the zigzag edge to indicate the presence of some edges between the two sets of vertices

Case-2: Suppose G has two vertices x_1 and x_2 of label $l \in \{l_1, l_2\}$ (see Figure 4b) and $\bar{l} \in \{l_1, l_2\} \setminus l$ such that x_1 (x_2) is universal to all vertices of label $l' \in \{l_1, l_2\}$ ($\bar{l}'$), but not adjacent to all vertices of label $\bar{l}'$ (l'). Then the algorithm relabels vertices x_1 and x_2 with l_3 and l_4 respectively and removes edges $E_{l_4, \bar{l}'}$, $E_{l_3, l'}$ to get the decomposition of G with $\rho_{l_4 \to \bar{l}}$, $\eta_{l_4, \bar{l}'}$, $\rho_{l_3 \to l}$, $\eta_{l_3, l'}$ above a $\oplus$ operation with children x_1, x_2 and connected components $G_{g_1}, \cdots, G_{g_k}$ ($G_g = x_1 \oplus x_2 \oplus G_{g_1} \oplus \cdots \oplus G_{g_k}$). If $G \cong_f^\pi H$, the algorithm finds vertices y_1 and y_2 in H of label $m \in \{l_1', l_2'\}$ and $\bar{m} \in \{l_1', l_2'\} \setminus m$ such that y_1 (y_2) is universal to all vertices of label m' ($\bar{m}'$), but not adjacent to all the vertices of label $\bar{m}'$

(m') and $f(x_1) = y_1$, $f(x_2) = y_2$. Then algorithm relabels vertices y_1 and y_2 with l'_3 and l'_4 respectively and removes edges $E_{l'_4, \bar{m}'}, E_{l'_3, m'}$ to get the decomposition of H with top operations $\rho_{l'_4 \to \bar{m}}, \eta_{l'_4, \bar{m}'}, \rho_{l'_3 \to m}, \eta_{l'_3, m'}$ above a $\oplus$ operation whose children are y_1, y_2 and connected components $H_{h_1}, \cdots, H_{h_k}$ (these are images of $G_{g_1}, \cdots, G_{g_k}$ under f in some order). The quotient graphs Q_g and Q_h build from the top operations are isomorphic via π_i, where $\pi_i(l') = \pi(l') = m'$, $\pi_i(\bar{l}') = \pi(\bar{l}') = \bar{m}'$, $\pi_i(l_3) = l'_3$ and $\pi_i(l_4) = l'_4$. It is clear that, $G_g \cong_f^{\pi_i} H_h$ and $\pi_i / color = \pi|_{color(Q_g)}$.

PC3: If $G \in PC3$ then G has two vertices x and y of label l, where y is universal to everything other than x, and x is universal to all vertices of label l other than y, and non-adjacent of all vertices of the other label $\bar{l}$ as shown in Figure 5. To decompose G the algorithm relabels the vertices x and y with l_3 and removes the edges $E_{l_3 l}$ to get the decomposition of G with top operations $\rho_{l_3 \to l}, \eta_{l_3, l}$ and a $\oplus$ with the connected components of $G_g = x \oplus G \setminus \{x\}$ as children. Again the algorithm removes the edges $E_{l_3 \bar{l}}$ from $G \setminus \{x\}$ to get the decomposition with top operations $\eta_{l_3, \bar{l}}$ and a $\oplus$ with the connected components of $G_{g_1} = y \oplus G \setminus \{x, y\}$ as children. If $G \cong_f^{\pi} H$, the algorithm finds the vertices $x', y' \in H$ of label l' such that $f(x) = x'$, $f(y) = y'$ and $\pi(l) = l'$. where y' is universal to everything other than x', and x' is universal to all vertices of label l' other than y', and non-adjacent to all vertices of label $\bar{l}'$. Then it relabels vertices x' and y' with l'_3 and removes the edges $E_{l'_3 l'}$ to get the decomposition of H with top operations $\rho_{l'_3 \to l'}, \eta_{l'_3, l'}$ and a $\oplus$ with the connected components of $H_h = x' \oplus H \setminus \{x'\}$ as children. Again the algorithm removes the edges $E_{l'_3, \bar{l}'}$ from $H \setminus \{x'\}$ to get the decomposition with top operations $\eta_{l'_3, \bar{l}'}$ and a $\oplus$ with the connected components of $H_{h_1} = y' \oplus H \setminus \{x', y'\}$ as children. In this case the generated parse tree has two levels.

In first level the quotient graphs Q_g and Q_h build from top operations are isomorphic via π_{1i}, where $\pi_{1i}(l) = \pi(l)$ if $l \in \{l_1, l_2\}$, $\pi_{1i}(l_3) = l'_3$. It is clear that, $G_g \cong_f^{\pi_{1i}} H_h$ and $\pi_{1i} / color = \pi|_{color(Q_g)}$. In second level the quotient graphs Q_{g_1} and Q_{h_1} build from top operations are isomorphic via π_{2i}, where $\pi_{2i}(l) = \pi_{1i}(l)$ if $l \in \{l_1, l_2, l_3\}$, and $G_{g_1} \cong_f^{\pi_{2i}} H_{h_1}$, $\pi_{2i} / color = \pi_{1i}|_{color(Q_{g_1})}$. The remaining part of proof follows from Lemma 13 and 14. $\qquad \square$

8.3 Function Decompose $\mathcal{U}_{l_1}$:

We next describe the case $\mathcal{U}_{l_1}$. The case $\mathcal{U}_{l_2}$ is omitted from here because it is similar to $\mathcal{U}_{l_1}$. Let l_1 and l_2 be the vertex labels. The vertex set V_{l_1} consisting of vertices with label l_1 can be partitioned as follows: The set of vertices adjacent to all vertices of V_{l_2} is denoted $V_{l_1}^a$. The set of vertices adjacent to some of vertices of V_{l_2} is denoted $V_{l_1}^s$. The set of vertices adjacent to none of vertices of V_{l_2} is denoted $V_{l_1}^n$. Similarly we can define the sets $V_{l_2}^a$, $V_{l_2}^s$ and $V_{l_2}^n$. For $l \in \{l_1, l_2\}$ we say, $G \in \mathcal{U}_l$ if $V_l^a \neq \emptyset$ and removing the edges between V_l^a and V_l disconnects G.

Lemma 13. *Let G and H be bilabeled, l-prime and connected graphs. If $G \cong_f^{\pi} H$ for some f and π then Algorithm 5 generates top operations of parse trees G*
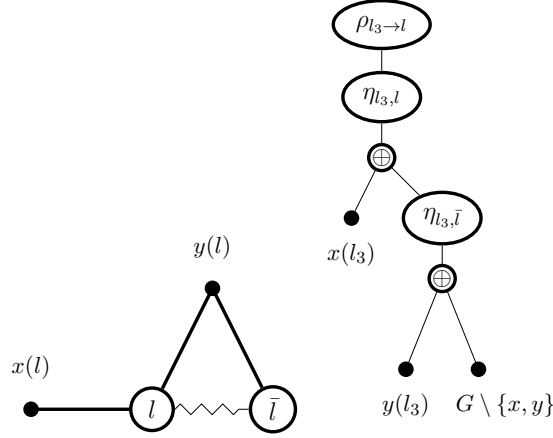
Fig. 5 Decomposing a bilabeled graph G which has two vertices x and y of label l, where y is universal to everything other than x, and x is universal to all vertices of label l other than y, and non-adjacent of all vertices of the other label $\bar{l}$

and H such that there is a $\pi_i \in \mathsf{ISO}(Q_g, Q_h)$ with $G_g \cong_f^{\pi_i} H_h$ and $\pi_i/color = \pi|_{color(Q_g)}$, where G_g and H_h are the graphs described in Algorithm 5.

Proof. $G \in \mathcal{U}_{l_1}$ if $V_{l_1}^a \neq \emptyset$ and removing the edges between $V_{l_1}^a$ and V_{l_2} disconnects G. The proof is divided into two cases based on connected components (partial[5] and non partial) of V_1.

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[5] A connected component of V_1 that contains at least one vertex of V_1^s is called partial.

Algorithm 5: Function *Decompose-leaf-$\mathcal{U}_{l_1}$* [6]

Input: A bilabeled, l-prime and connected graph G
Output: true with top operations of parse tree or false if $cwd(G) > 3$

1 **begin**
2   tree := null
3   **if** *G has good non partial connected component C* **then**
4   Let $G_g = \oplus_{i=1}^k G_{g_i}$, where G_{g_i}'s are connected components of $G \setminus \{E_{l_3, l_2}\}$
5         $tree = \rho_{l_3 \to l_1} \eta_{l_3, l_2}(\oplus_{i=1}^k G_{g_i})$
6   return (true, tree)
7   **if** *G has only partial connected components* **then**
8   Let $G_g = \oplus_{i=1}^k G_{g_i}$, where G_{g_i}'s are connected components of $G \setminus \{E_{l_3, l_2}\}$
9         $tree = \rho_{l_3 \to l_1} \eta_{l_3, l_2}(\oplus_{i=1}^k G_{g_i})$
10   return (true, tree)
11   return (false, tree) (i.e., $cwd(G) > 3$)

If there is at least one *good connected component*[6] C (see Section 5.2.1 in [6]) in G then the algorithm relabels all vertices of $V_{l_1}^a$ in good connected components with l_3 and removes the edges $E_{l_3 l_2}$ from G to get the decomposition with top operations $\rho_{l_3 \to l_1}$ and η_{l_3, l_2} above a $\oplus$ operation with the connected components $G_{g_1}, \cdots, G_{g_k}$ as children ($G_g = G_{g_1} \oplus \cdots \oplus G_{g_k}$). If $G \cong_f^\pi H$, up to a permutation of labels H may be in $\mathcal{U}_{l_1'}$ or $\mathcal{U}_{l_2'}$, but this does not effect the structure of the decomposition as in both the cases the set of edges deleted are same. The algorithm finds at least one good connected component C' in H and relabels all vertices of $V_{l_1'}^a$ in good connected components with l_3' and removes the edges $E_{l_3' l_2'}$ from H to get the decomposition with top operations $\rho_{l_3' \to l_1'}$ and $\eta_{l_3', l_2'}$ above a $\oplus$ operation with connected components $H_{h_1}, \cdots, H_{h_k}$ as children (these are images of $G_{g_1}, \cdots, G_{g_k}$ under f in some order). The quotient graphs Q_g and Q_h build from top operations are isomorphic via π_i, where $\pi_i(l) = \pi(l)$ if $l \in \{l_1, l_2\}$, $\pi_i(l_3) = l_3'$. It is clear that, $G_g \cong_f^{\pi_i} H_h$ and $\pi_i / color = \pi|_{color(Q_g)}$.

If there are only *partial components*[5] (see Section 5.2.1 in [6]) in graph G then the algorithm relabels all the vertices $V_{l_1}^a$ with l_3 and removes the edges $E_{l_3 l_2}$ from G to get the decomposition with top operations $\rho_{l_3 \to l_1}$ and η_{l_3, l_2} above a $\oplus$ operation with the connected components $G_{g_1}, \cdots, G_{g_k}$ as children ($G_g = G_{g_1} \oplus \cdots \oplus G_{g_k}$). If $G \cong_f^\pi H$, the algorithm relabels all the vertices $V_{l_1'}^a$ in H with l_3' and removes the edges $E_{l_3' l_2'}$ to get the decomposition with top operations $\rho_{l_3' \to l_1'}$ and $\eta_{l_3', l_2'}$ above a $\oplus$ operation with the connected components $H_{h_1}, \cdots, H_{h_k}$ as children (these are images of $G_{g_1}, \cdots, G_{g_k}$ under f in some order). The quotient graphs Q_g and Q_h build from top operations are isomorphic via π_i, where $\pi_i(l) = \pi(l)$ if $l \in \{l_1, l_2\}$, $\pi_i(l_3) = l_3'$, and $G_g \cong_f^{\pi_i} H_h$, $\pi_i / color = \pi|_{color(Q_g)}$.

Lemma 30, 31 in [6] shows that if $G \in \mathcal{U}_{l_1}$ apart from above two ways there is no other way to continue to find the decomposition for graphs of clique-width at most three. □

8.4 Function Decompose $\overline{\mathcal{D}}_{l_1}$:

Let V_l be the set of vertices with label l. For $l \in \{l_1, l_2\}$ we say, $G \in \overline{\mathcal{D}}_l$ if $\overline{V_l}$ is not connected and removing edges between the coconnected components of V_l disconnects G.

Lemma 14. *Let G and H be bilabeled, l-prime and connected graphs. If $G \cong_f^\pi H$ for some f and π then Algorithm 6 generates top operations of parse trees G and H such that there is a $\pi_i \in \mathsf{ISO}(Q_g, Q_h)$ with $G_g \cong_f^{\pi_i} H_h$ and $\pi_i / color = \pi|_{color(Q_g)}$, where G_g and H_h are the graphs described in Algorithm 6.*

Proof. The proof is divided into three cases depending on the structure of the graph.

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[6] A non-partial connected component C of V_1 is good (respectively, bad), if G is of clique-width at most three implies that the bilabeled graph obtained from C by relabeling all the vertices of $V_1^a \cup C$ with three is of clique-width at most three (respectively of clique-width more than three).

Algorithm 6: Function $Decompose\text{-}leaf\text{-}\overline{\mathcal{D}}_{l_1}$ [6]

Input: A bilabeled, l-prime and connected graph G
Output: true with top operations of parse tree or false if $cwd(G) > 3$

1 **begin**
2   tree := null
3   **if** *G has only two coconnected components* **then**
4   Let $G_g = \oplus_{i=1}^{k} G_{g_i}$, where G_{g_i}'s are connected components of $G \setminus \{E_{l_3 l_1}\}$
5         $tree = \rho_{l_3 \to l_1} \eta_{l_3, l_1} (\oplus_{i=1}^{k} G_{g_i})$
6   return (true, tree)
7   **if** *G has proper partition* **then**
8   Let $G_g = G_{g_1} \oplus G_{g_2}$, where G_{g_1} and G_{g_2} are connected components of $G \setminus \{E_{l_3 l_1}\}$
9         $tree = \rho_{l_3 \to l_1} \eta_{l_3, l_1} (G_{g_1} \oplus G_{g_2})$
10   return (true, tree)
11   **if** *G is eligible* **then**
12   Let $G_g = y \oplus (G_{g_1} \setminus y) \oplus_{i=2}^{k} G_{g_i}$, where G_{g_i}'s are connected components of $G \setminus \{E_{l_3 l_1}\}$ and $y, G_{g_1} \setminus y$'s are connected components of $G_{g_1} \setminus \{E_{l_3 l_2}\}$
13         $tree = \rho_{l_3 \to l_1} \eta_{l_3, l_1} (\eta_{l_3 l_2} (y \oplus G_{g_1} \setminus y) \oplus_{i=2}^{k} G_{g_i})$
14   return (true, tree)
15   return(false,tree) (i.e., $cwd(G) > 3$)

If there are only two coconnected components CCC_1 and CCC_2 of V_{l_1}, then the algorithm relabels one of CCC_1 or CCC_2 at random width l_3 and removes the edges $E_{l_3 l_1}$ to get the decomposition with top operations $\rho_{l_3 \to l_1}$ and η_{l_3, l_1} above a $\oplus$ operation with the connected components $G_{g_1}, \cdots, G_{g_k}$ as children $(G_g = G_{g_1} \oplus \cdots \oplus G_{g_k})$. If $G \cong_f^{\pi} H$, up to a permutation of labels H may be in $\overline{\mathcal{D}}_{l'_1}$ or $(\overline{\mathcal{D}}_{l'_2})$, without loss of generality assume H is in $\overline{\mathcal{D}}_{l'_1}$. In H the algorithm relabels one of the coconnected component of $V_{l'_1}$ at random with l'_3 and removes the edges $E_{l'_3 l'_1}$ to get the decomposition with top operations $\rho_{l'_3 \to l'_1}$ and $\eta_{l'_3, l'_1}$ above a $\oplus$ operation with the connected components $H_{h_1}, \cdots, H_{h_k}$ as children (these are images of $G_{g_1}, \cdots, G_{g_k}$ under f in some order). The quotient graphs Q_g and Q_h build from top operations are isomorphic via π_i, where $\pi_i(l_2) = l'_2$, $\pi_i(l_1) = l'_1$ or l'_3, $\pi_i(l_3) = l'_3$ or l'_1, and $G_g \cong_f^{\pi_i} H_h$, $\pi_i/color = \pi|_{color(Q_g)}$.

If G has a *proper partition*[7] then algorithm relabels one side of V_{l_1} with l_3 and removes the edges $E_{l_3 l_1}$ to get the decomposition with top operations $\rho_{l_3 \to l_1}$ and η_{l_3, l_1} above a $\oplus$ operation with the connected components G_{g_1} and G_{g_2} $(G_g = G_{g_1} \oplus G_{g_2})$ as children. If $G \cong_f^{\pi} H$, the algorithm relabels one side of $V_{l'_1}$ with l'_3 and removes the edges $E_{l'_3 l'_1}$ to get the decomposition with top operations $\rho_{l'_3 \to l'_1}$ and $\eta_{l'_3, l'_1}$ above a $\oplus$ operation with the connected components H_{h_1} and H_{h_2} as children. The quotient graphs Q_g and Q_h build from top operations are isomorphic via π_i, where $\pi_i(l_2) = l'_2$, $\pi_i(l_1) = l'_1$ or l'_3, $\pi_i(l_3) = l'_3$ or l'_1, and $G_g \cong_f^{\pi_i} H_h$, $\pi_i/color = \pi|_{color(Q_g)}$.

If G is *eligible* (see Section 5.2.2 in [6] for definition) then to decompose G the algorithm relabels vertices in coconnected component CCC_d with l_3 and

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[7]   partition of the coconnected components of V_{l_1} into two sides such that the vertices of V_{l_2} also partitions into two sides but no connected component of V_{l_2} has vertices in both sides

removes the edges $E_{l_3 l_1}$ to get the decomposition with top operations $\rho_{l_3 \to l_1}$, η_{l_3, l_1} and a $\oplus$ with the connected components $G_{g_1}, \cdots, G_{g_k}$ as children ($G_g = G_{g_1} \oplus \cdots \oplus G_{g_k}$). Again the algorithm removes the edges $E_{l_3 l_2}$ from G_{g_1} to get the decomposition with top operations η_{l_3, l_2} and a $\oplus$ with the connected components of $G_{g_1} = G_{g_1} \setminus \{y\} \oplus y$ as children. If $G \cong_f^\pi H$, the algorithm relabels coconnected component CCC'_d with l'_3 and removes the edges $E_{l'_3 l'_1}$ from H to get the decomposition with top operations are $\rho_{l'_3 \to l'_1}$, $\eta_{l'_3, l'_1}$ and a $\oplus$ with the connected components $H_{h_1}, \cdots, H_{h_k}$ (these are images of $G_{g_1}, \cdots, G_{g_k}$ under f in some order) as children. Again the algorithm removes the edges $E_{l'_3, l'_2}$ from H_{h_1} to get the decomposition with top operations $\eta_{l'_3, l'_2}$ and a $\oplus$ with the connected components of $H_{h_1} = H_{h_1} \setminus \{y'\} \oplus y'$ as children. In this case the generated parse tree has two levels. In the first level the quotient graphs Q_g and Q_h built from top operations are isomorphic via π_{1i}, where $\pi_{1i}(l) = \pi(l)$ if $l \in \{l_1, l_2\}$, $\pi_{1i}(l_3) = l'_3$, and $G_g \cong_f^{\pi_{1i}} H_h$, $\pi_{1i}/color = \pi|_{color(Q_g)}$. In the second level the quotient graphs Q_{g_1} and Q_{h_1} built from top operations are isomorphic via π_{2i}, where $\pi_{2i}(l) = \pi_{1i}(l)$ if $l \in \{l_1, l_2, l_3\}$, and $G_{g_1} \cong_f^{\pi_{2i}} H_{h_1}$, $\pi_{2i}/color = \pi_{1i}|_{color(Q_{g_1})}$.

Lemma 32 in [6] shows that if $G \in \overline{\mathcal{D}}_{l_1}$ apart from the above three ways there is no other way to continue to find the decomposition for graphs of clique-width at most three. $\qquad\square$